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On the Distribution Function and Moments of Power Sums With Log-Normal Components

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An approximate technique is presented for the evaluation of the mean and variance of the power sums with log-normal components. Exact expressions for the moments with two components are developed and then used in a nested fashion to obtain the moments of the desired sum. The results indicate more accurate estimates of these quantities over a wider range of individual component variances than any previously reported procedure. Coupling our estimates with the Gaussian assumption for the power sum provides a characterization of the cumulative distribution function which agrees remarkably well with a Monte Carlo simulation in the 1 to 99 percent range of the variate. Simple polynomial expressions obtained for the moments lead to an effective analytical tool for various system performance studies. They allow quick and accurate calculation of quantities such as cochannel interference caused by shadowing in mobile telephony.

I. INTRODUCTION

The power sum with K independent components

$$P_K = 10 \log_{10}\left[\sum_{k=1}^{K} 10^{X_k/10}\right] \tag{1}$$

is a random variable which appears in many areas of communications. With X_k Gaussian, the quantity

$$L_k = 10^{X_k/10} \tag{2}$$

is called a log-normal variate. The characterization of the sum of L_k is

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of importance in multihop scatter systems,[1] log-normal shadowing environments,[2,3] target detection in clutter,[4,5] and the general problem of propagation through a turbulent medium.* Thus, the distribution and moments of P_K are quantities of considerable importance. Unfortunately, these quantities do not appear to be expressible in simple analytical formulae and, as a consequence, approximate procedures have been investigated for some time. Of particular interest is the Wilkinson approach which uses a normal approximation for the distribution of P_K. The problem of characterizing the distribution function then reduces to finding the first two moments of the power sum.

The Wilkinson approach is consistent with an accumulated body of evidence indicating that, for the values of K that are of interest, the distribution of the sum of a finite number of log-normal random variables is well-approximated, at least to first-order, by another log-normal distribution.[1,9–12†] The central question, then, is how to estimate the mean and variance of the approximately Gaussian variate P_K, i.e., the power sum. Knowledge of the mean, m_x, and standard deviation, σ_x, of the Gaussian random variable X_k leads to a complete specification of the log-normal variate L_k, assuming, of course, no location parameter for the log-normal. Yet, the procedure for then estimating the moments of the power sum variable

$$P_K = 10 \log_{10}(L) = 10 \log_{10}(L_1 + L_2 + \cdots + L_K) \tag{3}$$

based on the individual moments m_x, σ_x, is by no means straightforward. As discussed in the next section, the Wilkinson approach leads to useful results only for a limited range of small values of the dB spread σ_x. Unfortunately, this is not the σ_x range of most practical interest.

The purpose of this paper is to outline a procedure that appears to be more accurate and to have a wider range of applicability than previously reported approximations. The new technique can be easily summarized: Analytical formulae are developed to compute the exact mean and variance for the power sum witb two components,

$$P_2 = 10 \log(L_1 + L_2). \tag{4}$$

We then suppose P_2 is Gaussian, i.e., it is assumed that $L_1 + L_2 = 10^{P_2/10}$ is log-normal. We then consider

$$P_3 = 10 \log(L_1 + L_2 + L_3) = 10 \log(10^{P_2/10} + L_3) \tag{5}$$

and compute the mean and variance using the derived formulae. In this manner, we iterate until the required moments of P_K are obtained.

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* Many more applications, including the field of economics and the estimation of crude oil reserves, are noted in Refs. 6, 7, and 8.

† We also confirm this observation here by means of Monte Carlo simulations.

We have compared the results of a Monte Carlo study to the estimates of moments obtained from the new method just described, and permuted versions of it. Our procedure leads to remarkably accurate estimates for the mean of P_K, over a wide range of component variances. The estimate of the standard deviation of P_K is accurate for a somewhat more limited range but, nevertheless, a much wider range than for previously reported approximations.

The rest of this paper is organized as follows. In the next section, we establish additional notation and discuss, in more detail, previous work relating to power sums. The analytical development is presented in Section III, with the details reserved for the Appendix. The development of both accurate and simple expressions for the mean and variance of P_2 is of considerable importance for system performance studies in areas such as shadowing in mobile radio and log-normal fading in microwave radio. In Section III, we also present such an expression in the form of simple polynomials. Simulation results, validating both the Gaussian approximation for P_K and the new estimation technique, are presented in Section IV. Concluding comments are given in Section V.

II. PRELIMINARIES

The power sum, as defined in (1), is measured in decibels. However, rather than use the definition for L_k given in (2), it is more convenient to use the natural logarithm:

$$L_k = e^{Y_k}. \tag{6}$$

The relationship between the two associated normal variates is simply

$$Y_k = \lambda X_k, \tag{7}$$

where

$$\lambda = \tfrac{1}{10} \log_e 10 = 0.23026. \tag{8}$$

When the mean and variance of X_k are specified as, say, m_x and σ_x, in order to use the representation (6), we have the obvious scaling

$$m_y = \lambda m_x, \qquad \sigma_y^2 = \lambda^2 \sigma_x^2. \tag{9}$$

Similarly, we define the quantities

$$L = 10^{P_K/10} = \mathrm{e}^Z, \tag{10}$$

and the corresponding moments are related by

$$m_z = \lambda m_{P_K}, \qquad \sigma_z^2 = \lambda^2 \sigma_{P_K}^2. \tag{11}$$

The rth moment of a log-normal variate L is given in terms of the moment-generating function of Y:

$$E(L_1^r) = E(e^{rY_1}) = e^{rm_{y_1}+1/2(r^2\sigma_{y_1}^2)}. \tag{12}$$

Note that the moments grow exponentially with the order r^2. This rapid growth of moments may be one of the reasons for the limited range of applicability of the classical Wilkinson approximation (see Ref. 11); dealing with diffferences of large numbers can lead to numerical instability in trying to evaluate the approximate log-normal density of the sum.

Marlow[13] has shown that under quite general conditions, as $K \to \infty$, P_K is asymptotically normally distributed.* For fixed, finite K, and with the X_K independent, identically distributed normal random variables (with mean m_x and standard deviation σ_x), Marlow also derived a small variance normal approximation.[14] As $\sigma_x \to 0$, a scaled version of P_K approaches the distribution function of a unit normal.

In terms of log-normal variates, if we write

$$P_K = 10 \log_{10}(L) = 10 \log_{10}(L_1 + L_2 + \cdots + L_K), \tag{13}$$

Marlow's second result says that, in the finite-component, small-variance case, the sum of log-normal variates is also approximately log-normal. This result forms the analytical basis for the classical Wilkinson approximation, which is the usual normal approximation for the distribution of the power sum.

The Wilkinson approximation proceeds as follows: With

$$L = e^Z, \tag{14}$$

now taken to be log-normal, Z is the associated Gaussian random variable with parameters m_z and σ_z. To find these quantities, one uses (12) and equates the first two moments of both sides of (14). For example, with $K = 2$, and Y_1, and Y_2 identically distributed with mean m_y and variance σ_y^2,

$$E(L) = e^{m_z+1/2\sigma_z^2} = 2e^{m_y+1/2\sigma_y^2}$$

$$E(L^2) = e^{2m_z+2\sigma_z^2} = 2e^{2m_y+2\sigma_y^2} + 2(e^{m_y+1/2\sigma_y^2})^2. \tag{15}$$

Taking logarithms gives a set of linear equations for the two unknowns, m_z and σ_z^2.

Figure 3, which will be discussed in detail later, presents the results of a Monte Carlo simulation and the Wilkinson approximation, employing eq. (15). As indicated, the Wilkinson approach tends to break down for $\sigma_x = \sigma_y/\lambda$ greater than 4 dB. This is consistent with the above-quoted result from Marlow.

*For the infinite component case, Marlow also points out that there is asymptotic normality both on the power scale, as well as on the dB scale. That is, both P_K and L are asymptotically normal.

If one is interested in small σ, but larger values of the random variable, i.e., the tails of the power sum distribution, Fenton[1] suggests a better approximation to the equivalent log-normal distribution by utilizing higher-order moments. In this case, eq. (15) is replaced with equality of, say, third and fourth moments. This procedure of using different values of r in different regions results in an approximating distribution which would be a series of connected straight lines when plotted on log-probability graph paper.

At the other end of the σ_x component range, Farley[9] has derived a large variance approximation. With independent, identically distributed variables, as $\sigma_x \to \infty$

$$Pr\left[\frac{1}{\sigma_x}(P_K - m_x) < \alpha\right] = [\Phi(\alpha)]^K, \tag{16}$$

where $\Phi(\alpha)$ is the unit normal distribution function. (Clearly, this is a decidedly non-Gaussian result.) Farley's simulations indicate that the large variance approximation gives better results than Wilkinson's approach for about $\sigma_x > 10$ dB and K fixed.

In this study, we also take K finite, but focus on the midrange, $4 \leq \sigma_x \leq 12$ dB, where neither Marlow's nor Farley's results are applicable. This midrange is of particular interest for the shadowing phenomena in mobile telephony, microwave radio fading, and airborne radar clutter.

III. ANALYTICAL RESULTS

We assume that

$$L = \sum_{k=1}^{K} L_k = \sum_{k=1}^{K} e^{Y_k} = e^Z \tag{17}$$

for finite K is log-normal. The associated Gaussian variate is Z, and the goal is to determine the quantities

$$\begin{aligned} m_z &= E(\ln L) \\ \sigma_z^2 &= E[(\ln L - m_z)^2] \end{aligned} \tag{18}$$

in terms of the mean and standard deviation of the component variates Y_k.

Below, we give exact expressions for m_z and σ_z when there are two components. The procedure for K greater than two combines the log-normal variates in a nested fashion two-by-two, using the exact formulas developed. To illustrate the procedure, suppose we have three log-normal variates: $L = L_1 + L_2 + L_3$. The exact first and second moments of $\ln(L_1 + L_2)$ are computed. We then take

$$Z_2 = \ln(L_1 + L_2) \tag{19}$$

as a normal random variable and write*

$$Z = \ln(L) = \ln(e^{Z_2} + L_3) = \ln(e^{Z_2} + e^{Y_3}). \tag{20}$$

We then compute the moments of $\ln(L)$, again using the exact expressions developed for two components. This procedure has been tested in a variety of situations and groupings of variates. The results are very encouraging and will be discussed in the next section. Here, we outline the development of the analytical formulae, with the details in the Appendix.

Naus[15] has derived the moment-generating function, and computed the first two moments of the power sum with two independent, identically distributed normal components. (Hamdan[16] generalized Naus' result for correlated variables with unequal variances.) Here, we extend Naus' general approach to the case of two independent variables with unequal means and variances.

For the two components, we have

$$e^Z = L = L_1 + L_2 = e^{Y_1} + e^{Y_2} \tag{21}$$

or

$$Z = \ln(e^{Y_1} + e^{Y_2}).$$

Define the Gaussian random variable

$$w = Y_2 - Y_1 \tag{22}$$

$$m_w = \bar{w} = m_{y_2} - m_{y_1} \tag{23}$$

$$\sigma_w^2 = \sigma_{y_2}^2 + \sigma_{y_1}^2. \tag{24}$$

Taking the expectation of Z, we have

$$\begin{aligned} E\{Z\} &= E[\ln(e^{Y_1} + e^{Y_2})] \\ &= E\{\ln[e^{Y_1}(1 + e^{Y_2 - Y_1})]\} \\ &= E(Y_1) + E[\ln(1 + e^w)]. \end{aligned} \tag{25}$$

The second term is

$$E[\ln(1 + e^w)] = \int_{-\infty}^{+\infty} [\ln(1 + e^w)]f(w)dw, \tag{26}$$

where $f(w)$ denotes the normal density with the above-indicated mean and variance. The logarithmic term is now expanded in a power series. To ensure convergence of both the power series and the series resulting from the subsequent integration, the integral is broken up into the appropriate ranges. Thus,

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* Equivalently, $L_1 + L_2$ is assumed to be log-normal.

$$\int_{-\infty}^{+\infty} \ln(1 + e^{w}) f(w)\, dw = \int_{-\infty}^{0} \ln(1 + e^{w}) f(w)\, dw + \int_{0}^{\infty} [\ln(1 + e^{-w}) + w] f(w)\, dw. \quad (27)$$

The expansion

$$\ln(1 + x) = \sum_{j=1}^{\infty} C_j x^j, \; C_j = \frac{(-1)^{j+1}}{j}, \quad (28)$$

valid for $|x| < 1$, can now be used for each of the above integrals. The result (details are given in the Appendix) for the first moment is

$$\begin{aligned}
m_z &= E[\ln(L_1 + L_2)] \\
&= m_{y_1} - \frac{\sigma_w}{\sqrt{2\pi}} e^{-m_w^2/2\sigma_w^2} + m_w \, \Phi\left(\frac{m_w}{\sigma_w}\right) \\
&\quad + \sum_{k=1}^{\infty} C_k e^{k^2 \sigma_w^2/2} \left[e^{k m_w} \, \Phi\left(\frac{-m_w - k\sigma_w^2}{\sigma_w}\right) \right. \\
&\quad \left. + \, c^{-k m_w} \, \Phi\left(\frac{m_w - k\sigma_w^2}{\sigma_w}\right) \right] \\
&= m_{y_1} + G_1(\sigma_w, m_w),
\end{aligned} \quad (29)$$

where $\Phi(x)$ is the Gaussian probability distribution function as defined in the Appendix, eq. (39), and G_1 is defined implicitly. It is not difficult to show that the series in (29) converges. Our computer experience indicates that about 40 terms are required for the 4th significant digit.

As might be expected, the formula for the second moment is more complicated and is given by eq. (72) in the Appendix. The basic idea however, remains the same: When expansions of logarithmic functions are used, the expectation is broken up into integration ranges which provide convergent series.

Although the formulae as developed involve infinite series and are complicated, their application is relatively simple. We can write [see eqs. (49) and (73) in the Appendix]:

$$\begin{aligned}
m_z &= m_{y_1} + G_1(\sigma_w, m_w) \\
\sigma_z^2 &= \sigma_{y_1}^2 - G_1^2(\sigma_w, m_w) - 2\rho^2 G_3(\sigma_w, m_w) + G_2(\sigma_w, m_w),
\end{aligned} \quad (30)$$

where the $G_i(\sigma_w, m_w)$ are defined in the Appendix and

$$\rho = -\sigma_{y_1}/\sigma_w. \quad (31)$$

Observe that the G_i functions, $i = 1, 2, 3$, depend only on the two parameters m_w, σ_w. Consequently, we can evaluate $G_i(\sigma_w, m_w)$ as a

Table Ia—Coefficients of the approximating polynomials—Region I
$[-20 < m_w < 0, 0 < \sigma_w < 15]$

Coefficients	Functions G1	G2	G3
A00	−0.1153239E 00	0.4012876E-01	−0.3958699E 01
A01	−0.5667912E 01	−0.4483259E 01	−0.5454983E 01
A02	0.1151279E 02	0.7391760E 01	0.1139280E 02
A03	−0.7162489E 01	−0.3772190E 01	−0.7116366E 01
A04	0.1312986E 01	0.5262268E 00	0.1315218E 01
A10	−0.1611385E 00	−0.1579114E 01	0.6839918E 01
A11	0.2084215E 02	0.1637249E 02	0.1962529E 02
A12	−0.4499768E 02	−0.2994901E 02	−0.4314091E 02
A13	0.2756210E 02	0.1337812E 02	0.2647795E 02
A14	−0.5109783E 01	−0.1747459E 01	−0.4940592E 01
A20	0.1345124E 00	0.2174588E 01	−0.4717296E 01
A21	−0.2670183E 02	−0.2114152E 02	−0.2486802E 02
A22	0.5919191E 02	0.3976749E 02	0.5618572E 02
A23	−0.3695334E 02	−0.1776890E 02	−0.3501815E 02
A24	0.6912766E 01	0.2195322E 01	0.6560938E 01
A30	0.8057054E-01	−0.7525302E 00	0.1819360E 01
A31	0.1429709E 02	0.1147157E 02	0.1323538E 02
A32	−0.3225969E 02	−0.2189501E 02	−0.3050022E 02
A33	0.2055628E 02	0.1006767E 02	0.1937757E 02
A34	−0.3888684E 01	−0.1224983E 01	−0.3658379E 01
A40	−0.3145306E-01	0.8447987E-01	−0.2817493E 00
A41	−0.2730047E 01	−0.2222839E 01	−0.2518272E 01
A42	0.6244253E 01	0.4289557E 01	0.5893337E 01
A43	−0.4048245E 01	−0.2035757E 01	−0.3809492E 01
A44	0.7746786E 00	0.2499568E 00	0.7260144E 00

function of these two parameters and store the results in three lookup tables. An alternative is suggested by Fig. 5, which gives the three G_i's as a function of σ_w with m_w as a parameter. Since the curves are smooth, we can fit low-order polynomials to them and use the resulting expressions in (30) for the evaluation of the moments m_z and σ_z. The advantage of utilizing these analytical expressions in system performance studies is clear and, indeed, is what we have implemented.

A least-squares fit was performed to determine the polynomial coefficients. After some experimentation, we chose the polynomial

$$\log_{10} G_i(\sigma_w, m_w) = \sum_{j=0}^{J} \sum_{k=0}^{K} A_{jk}(i) \sigma_w^{j/2} \left| m_w \right|^{k/2}, \qquad i = 1, 2, 3, \tag{32}$$

with $J = K = 4$. The coefficients $A_{jk}(i)$ were obtained for two regions of the parameter space. The results for Region I ($-20 \le m_w \le 0$, $0 \le \sigma_w \le 15$) and Region II ($-40 \le m_w \le -20$, $0 \le \sigma_w \le 15$) are presented in Tables Ia and Ib. Over these regions of interest, the maximum error produced by using the polynomial approximations was found to be about 1 percent as compared to the exact calculations based on eqs. (49) and (73).

Before discussing the results of the simulation study, we digress to note how the analytical results can be extended. First, the development

Table Ib—Coefficients of the approximating polynomials—Region II

$[-40 < m_w < -20, 0 < \sigma_w < 15]$

	Functions		
Coefficients	G1	G2	G3
A00	−0.3354792E 03	0.8942737E 02	−0.3596955E 03
A01	0.5281230E 03	−0.1521976E 03	0.5576596E 03
A02	−0.3085414E 03	0.9734654E 02	−0.3242909E 03
A03	0.7895054E 02	−0.2744724E 02	0.8271772E 02
A04	−0.7518427E 01	0.2756644E 01	−0.7863267E 01
A10	0.1354559E 04	−0.3688427E 03	0.1443269E 04
A11	−0.2133838E 04	0.6287175E 03	−0.2250748E 04
A12	0.1245491E 04	−0.4077784E 03	0.1307118E 04
A13	−0.3191052E 03	0.1148104E 03	−0.3336339E 03
A14	0.3036738E 02	−0.1161187E 02	0.3167827E 02
A20	−0.1865732E 04	0.4966108E 03	−0.1986007E 04
A21	0.2941369E 04	−0.8514775E 03	0.3106047E 04
A22	−0.1718295E 04	0.5574036E 03	−0.1804377E 04
A23	0.4403198E 03	−0.1588548E 03	0.4603419E 03
A24	−0.4188256E 02	0.1624075E 02	−0.4366067E 02
A30	0.1059090E 04	−0.2684747E 03	0.1128759E 04
A31	−0.1671264E 04	0.4638312E 03	−0.1768423E 04
A32	0.9776851E 03	−0.3061915E 03	0.1028409E 04
A33	−0.2507169E 03	0.8842364E 02	−0.2624284E 03
A34	0.2384698E 02	−0.9152624E 01	0.2487490E 02
A40	−0.2121206E 03	0.5077676E 02	−0.2264791E 03
A41	0.3350888E 03	−0.8833020E 02	0.3553536E 03
A42	−0.1963238E 03	0.5876009E 02	−0.2069294E 03
A43	0.5040234E 02	−0.1717538E 02	0.5284466E 02
A44	−0.4795980E 01	0.1799387E 01	−0.5008795E 01

is valid when the random variables Y_1 and Y_2 are correlated as well as having different means and variances. With a simple modification in eq. (24) to account for the correlation [and another in (52)],

$$\sigma_w^2 = \sigma_{y_2}^2 + \sigma_{y_1}^2 - \rho_{12}\sigma_{y_1}\sigma_{y_2}, \tag{33}$$

the results remain unchanged. Furthermore, in principle, the underlying component variables do not have to be Gaussian. The proviso, of course, is that the integration with the non-Gaussian $f(w)$ be tractable, and it must fall off fast enough so that the resulting series is convergent. (Clearly, the components are no longer log-normal.)

Finally, we observe that the results can be extended in yet another direction. By generalizing the procedure outlined above, exact formulae for moments of more than two components can be obtained. However, the resulting expressions are very complicated and, given the high accuracy of our new method, the added complexity does not appear to be warranted.

IV. SIMULATION RESULTS

We use the definitions introduced earlier:

$$L_k = 10^{X_k/10} \tag{34}$$

$$P_K = 10 \log_{10}(L) = 10 \log_{10}\left[\sum_{k=1}^{K} L_k\right]. \tag{35}$$

In tbe following discussion, by m_p and σ_p, we shall mean the first moment and standard deviation of tbe random variable P_K. The value of K should be clear from the context. We focused our Monte Carlo study on three issues:

(*i*) How good is the assumption tbat $L = L_1 + L_2 + \cdots + L_K$ is approximately log-normal or, equivalently, how well is tbe cumulative distribution function (cdf) of $P_K = \log_{10}(L)$ described by a Gaussian cdf? Furthermore, how closely does the Gaussian cdf (based on the calculated m_p and σ_p) match the true cdf of the power sum?

(*ii*) How accurate is the new analytical method in estimating the resultant mean m_p and standard deviation σ_p and for wbat range of dB spread σ_x and component number K is the technique accurate?

(*iii*) Is the nested procedure we have described numerically robust, or are the estimates of m_p and σ_p sensitive to the order in which we combine the log-normal components?

The results of our Monte Carlo simulation confirmed the observation of a number of other investigators, that the cdf of P_K is well approximated by the Gaussian cdf, particularly in the range of practical interest.* Shown in Fig. 1 is the cdf of the sum of two log-normal variates with $\sigma_x = 6$, 10, and 14 dB. We note that the Monte Carlo simulation agrees quite closely with the assumed Gaussian cdf (based on the calculated m_p and σ_p) in the range of 0.1 to 99 percent. Outside this range, the simulation values start to deviate slightly. The cdf of the sum of a large number of log-normal variates, discussed below in Examples 1 to 3, together with a fourth case (the sum of six equal components with $m_x = 0$ dB and $\sigma_x = 10$ dB), are presented in Fig. 2. Here we observe excellent agreement between the calculated (assumed) and simulated cdf in the range 1 to 99 percent.

Having confirmed the Gaussian approximation for P_K and the fact that the calculated cdf closely represents the true cdf, we turn our attention to the second issue, estimating the moments m_p and σ_p from the moments of the individual log-normal components. In Figs. 3a to 3c, the number of identically distributed components, K, is held fixed and the component variance (dB spread) σ_x^2 is varied. In all cases, the component mean m_x is 0. Figure 3a illustrates the results for two components and serves to verify the Monte Carlo program. Our computed results coincide (as they should) with the simulation. In contrast, the Wilkinson approximation begins to give inaccurate results

* The simulations typically had 10,000 sample points for each value quoted or indicated on the figures.

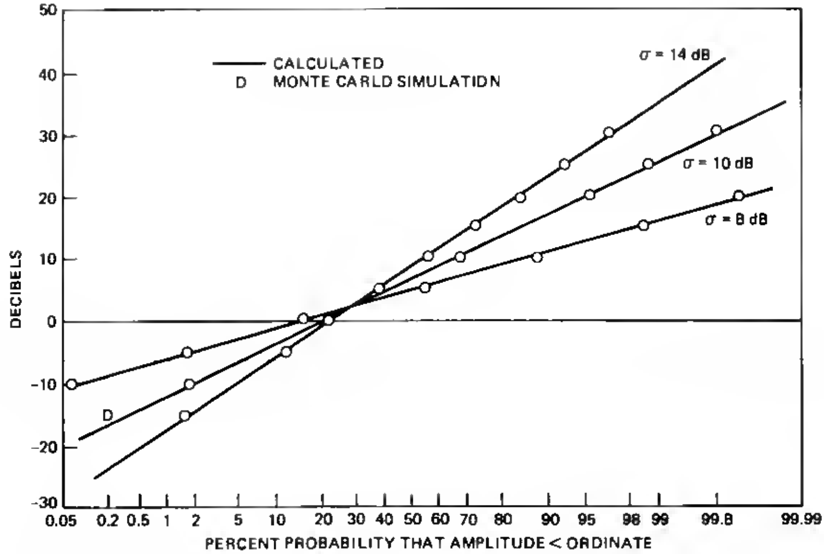
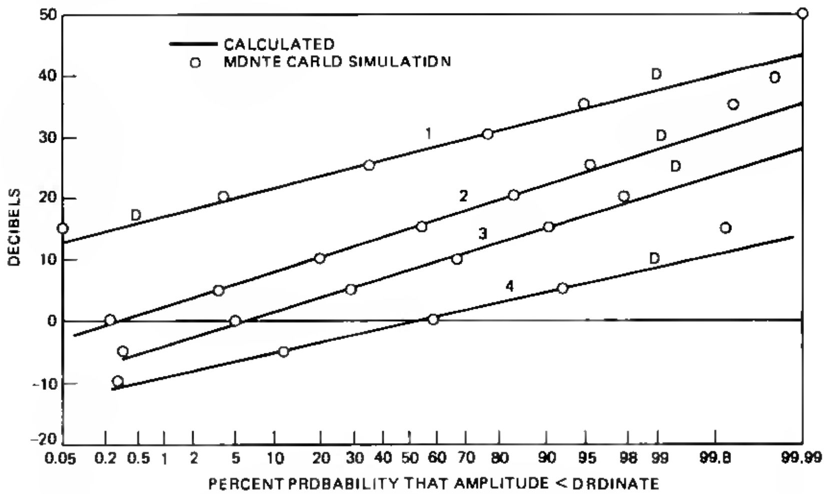


Fig. 1—Cumulative distribution function of sum of two log-normal variates with zero means and variances equal to 6, 10, and 14 dB.



1. THREE VARIATES: $m = 0$, $\sigma = 8, 7, 9.5$.
2. SIX EQUAL COMPONENTS: $m = 0$, $\sigma = 10$.
3. NINE COMPONENTS IN THREE GROUPS: GROUP 1, $m = -10$, $\sigma = 6$; GROUP 2, $m = -18$, $\sigma = 10$; GROUP 3, $m = -38$, $\sigma = 12$.
4. EIGHTEEN COMPONENTS IN THREE GROUPS: GROUP 1, $m = 10$, $\sigma = 10$; GROUP 2, $m = -2$, $\sigma = 10$; GROUP 3, $m = -8$, $\sigma = 10$.

Fig. 2—Cumulative distribution function of the sum of a large number of log-normal variates corresponding to Examples 1 to 3 in Section IV plus an additional case.

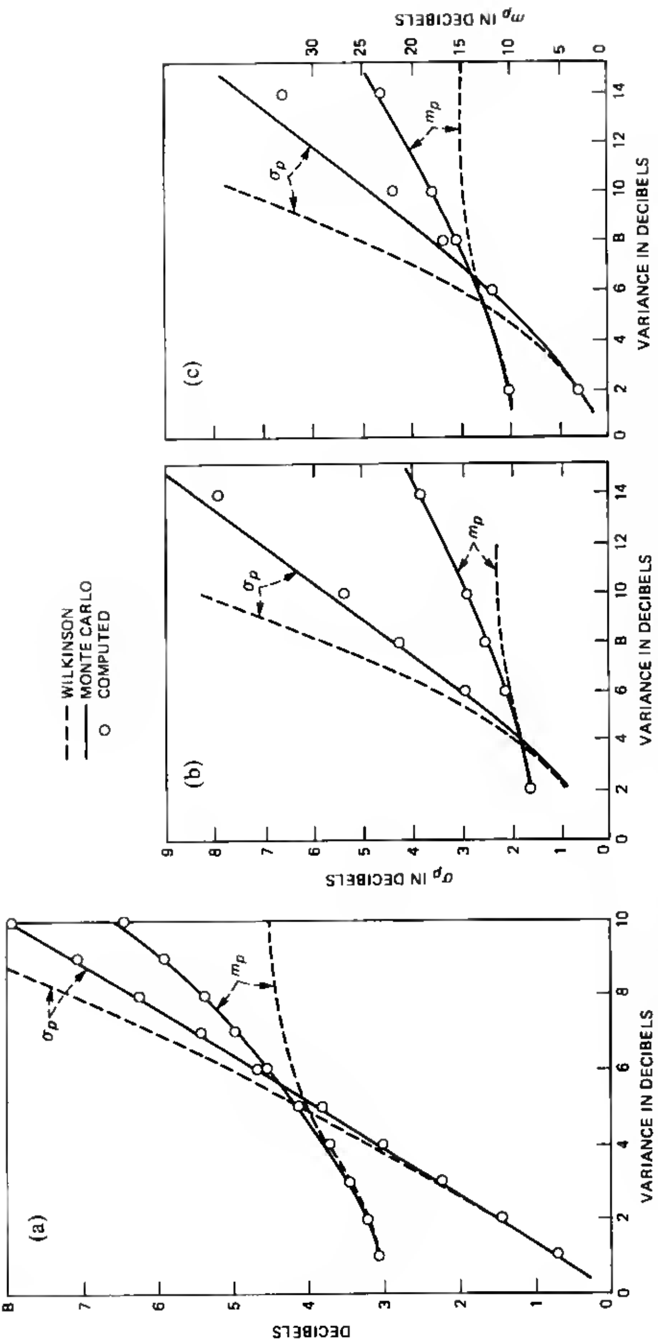


Fig. 3—Mean and standard deviation of power sum with (a) two, (b) 5:x, and (c) ten log-normal comonents as a function of comopnent standard deviation.

at about σ_x = 4 dB and becomes increasingly worse with increasing σ_x.

Figure 3b illustrates six components, and Fig. 3c illustrates ten. The dB spread ranges from 2 to 14 dB. Our technique gives remarkably accurate estimates of m_p, while the estimate of σ_p is quite close for $\sigma_x < 8$ dB and then tends to underestimate the true standard deviation. By way of contrast, for these ranges and number of components, the Wilkinson approach gives gross inaccuracies and cannot be used.

In Figs. 4a and 4b, we vary the number of components, set $m_x = 0$, and hold the dB spread fixed. Again, our procedure gives very accurate estimates of the mean and, as the number of components increases, the variance is underestimated only slightly for $\sigma_x = 6$ dB (Fig. 4a) and somewhat more so when $\sigma_x = 10$ dB (Fig. 4b). For these cases, the error in the estimate of the variance as a function of the number of components is as shown in Table II. (There are negligible errors in estimates of the mean.) As indicated in tbe figures, the errors in the Wilkinson approach are substantial, ranging up to 74 percent when $\sigma_x = 10$ dB.

It should be observed that the situation with equal mean components and large dB spread is probably a worst-case situation. Indeed, from the above table we see for $K = 16$, $\sigma_x = 10$ dB, that the estimate of σ_p is off by 12.8 percent. In contrast, in Example 3 below, we deal witb $K = 18$ components, and the same dB spread. There, the error in the estimate is less—about 6 percent. The only substantial difference in the two situations is the unequal mean values. These comments notwithstanding, further study is warranted for this boundary of parameter values, i.e., where the number of equal mean components is high ($K \geq 16$) and the dB spread is identical and large ($\sigma_x \geq 10$ dB).

Our technique was also tested on a number of examples with unequal components. We report on three that were chosen to illustrate the general applicability of the new method.

Example 1—Three components, equal means, different variances:

$$m_{x_i} = 0 \text{ dB}, \quad i = 1, 2, 3; \qquad \sigma_{x_1} = 6, \quad \sigma_{x_2} = 7, \quad \sigma_{x_3} = 9.5 \text{ dB}.$$

The Monte Carlo simulation gave $m_p = 8.08$, $\sigma_p = 5.356$. Our procedure provided the estimates $m_p = 8.05$, $\sigma_p = 5.273$, for a 0.03-dB error in the mean and 1.5 percent error in the variance.

Example 2—Nine components, taken in homogeneous groups of three:

$$m_{x_i} = -38 \text{ dB}; \qquad \sigma_{x_i} = 12 \text{ dB}, \quad i = 1, 2, 3$$

$$m_{x_i} = -18 \text{ dB}; \qquad \sigma_{x_i} = 10 \text{ dB}, \quad i = 4, 5, 6$$

$$m_{x_i} = -10 \text{ dB}; \qquad \sigma_{x_i} = 6 \text{ dB}, \quad i = 7, 8, 9.$$

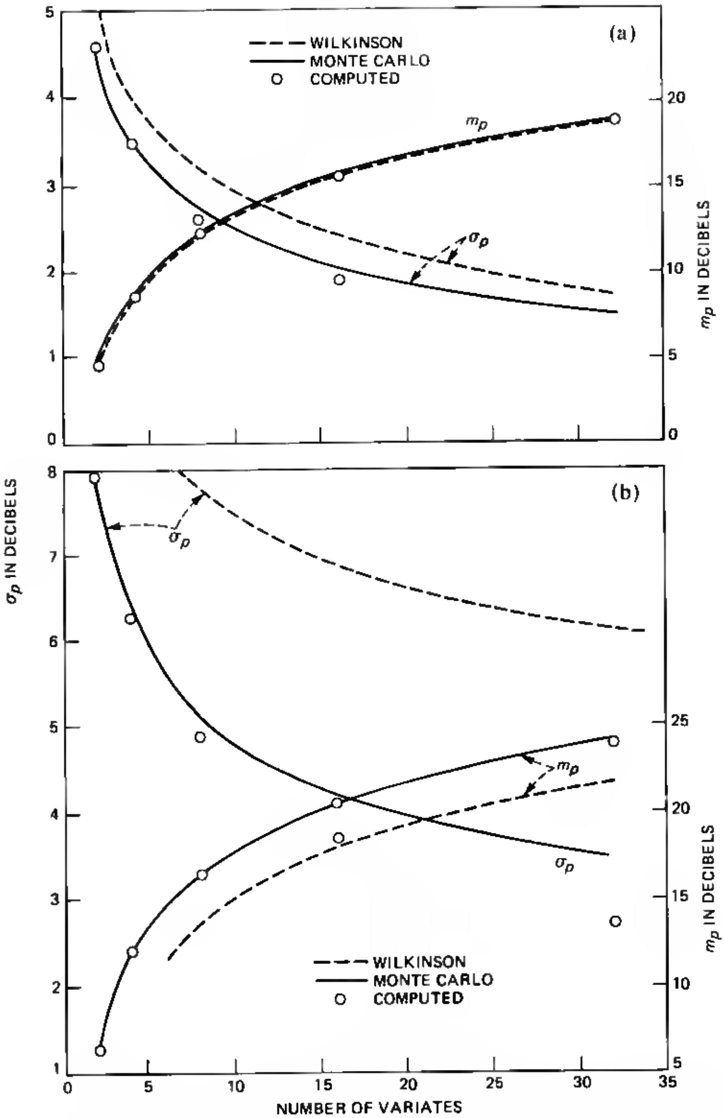


Fig. 4—Mean and standard deviation of power sum with varying number of identically distributed components. (a) Component mean, $m_x = 0$ and standard deviation, $\sigma_x = 6$ dB. (b) Component standard deviation $\sigma_x = 10$ dB.

Monte Carlo results gave $m_p = -0.61$, $\sigma_p = 3.90$, while our technique yielded $m_p = -0.6$, $\sigma_p = 3.79$. This gives an error of 0.01 dB in m_p and 2.8 percent in σ_p.

We now address the third issue raised at the beginning of this section. To show that the way in which we combine the variables has

Table II—Error in estimate of variance

	No. of Components (K)				
Percent Error	2	4	8	16	32
$\sigma_x = 10$ dB	0.13	1.4	5.4	12.8	21.4
$\sigma_x = 6$ dB	0.43	2.0	3.3	8.2	11.5

little effect, we permuted the order in combining the nine components. The resultant mean estimate varied between -0.59 and -0.64 dB, while the standard deviation varied between 3.66 and 3.89 dB. This example—and a number of others studied—illustrates the desired numerical robustness of the procedure.

Example 3—Eighteen components: This example would correspond to two tiers of interferers in a cellular mobile radio scheme.

Components 1 to 6: $m_x = 10$ dB, $\sigma_x = 10$ dB.
Components 7 to 12: $m_x = -2$ dB, $\sigma_x = 10$ dB.
Components 13 to 18: $m_x = -8$ dB, $\sigma_x = 10$ dB.

The Monte Carlo results are: $m_p = 27.07$, $\sigma_p = 4.54$, and our technique gave $m_p = 27.04$, $\sigma_p = 4.26$. The error is 0.03 dB in the mean estimate and 6.2 percent for σ_p.

V. CONCLUSIONS

In this study, we have verified once again, as others have, that the sum of a moderate number of log-normal random variables is well approximated by another log-normal variate, especially in the cdf range of 1 to 99 percent. Perhaps more important, we have been able to relate the mean and variance of the resultant (Gaussian) power sum to the first two moments of the individual underlying Gaussian components. Our method is highly accurate in the range of parameter values of most practical interest.

Based on our preliminary simulations, we may conclude that the analytical method presented in this paper is extremely accurate in evaluating the mean of the power sum for the complete range of parameter values investigated. This range was a dB spread of $2 \le \sigma_x \le 14$ dB and up to $K = 30$ components. It is also accurate in estimating the standard deviation with up to eight equal log-normal variates when σ_x is less than 10 dB. Outside this range, the approximation is slightly less accurate.

APPENDIX

Derivation of First and Second Moments

Our purpose is to provide enough detail so that an interested reader can rederive the formulae for the first and second moments, as given below by eqs. (48) and (73).

The expansions we use are

$$\ln(1+x) = \sum_{k=1}^{\infty} C_k x^k, \quad |x| < 1 \quad \text{and} \quad C_k = \frac{(-1)^{k+1}}{k} \tag{36}$$

$$\ln^2(1+x) = \sum_{k=1}^{\infty} b_k x^{k+1},$$

$$|x| < 1 \quad \text{and} \quad b_k = \frac{2(-1)^{k+1}}{k+1} \sum_{j=1}^{k} j^{-1}. \tag{37}$$

The density function of a normal random variable with arbitrary mean and variance is defined by

$$\Phi'\ (x;\, m,\, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}. \tag{38}$$

The distribution function is a simple parameter function and is given by

$$\Phi\left(\frac{x-m}{\sigma}\right) = \int_{-\infty}^{x} \Phi'(t;\, m,\, \sigma)dt \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-m}{\sigma}} e^{-t^2/2} dt. \tag{39}$$

The function $\Phi(x)$ is related to the error function by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{40}$$

$$\Phi(x) = ½ + ½ \operatorname{erf}(x/\sqrt{2}). \tag{41}$$

Three integrals which occur often in the development are

$$E\{e^{rx}\} = \int_{-\infty}^{+\infty} e^{rx} \Phi'(x;\, m,\, \sigma)dx = e^{mr + r^2\sigma^2/2} \tag{42}$$

$$\int_{-\infty}^{0} e^{rx} \Phi'(x;\, m,\, \sigma)dx = [e^{mr+r^2\sigma^2/2}]\Phi\left(\frac{-m - \sigma^2 r}{\sigma}\right)$$

$$\int_0^{\infty} e^{-rx} \Phi'(x;\, m,\, \sigma)dx = [e^{-mr+r^2\sigma^2/2}]\Phi\left(\frac{m - \sigma^2 r}{\sigma}\right). \tag{43}$$

We repeat the definitions introduced in Section III:

$$e^Z = L = L_1 + L_2 = e^{Y_1} + e^{Y_2}$$

$$Z = \ln(e^{Y_1} + e^{Y_2}),$$

with the Y_i Gaussian random variables. Let

$$w = Y_2 - Y_1$$

$$m_w = \bar{w} = E(w) = m_{y_2} - m_{y_1}$$

$$\sigma_w^2 = E(w - \bar{w})^2 = \sigma_{y_2}^2 + \sigma_{y_1}^2. \tag{44}$$

For convenience, we shall let $f(w)$ denote the particular Gaussian density function

$$f(w) = \Phi'(w; m_w, \sigma_w); \tag{45}$$

Z is rewritten as

$$Z = \ln(e^{Y_1} + e^{Y_2}) = \ln[e^{Y_1}(1 + e^w)]. \tag{46}$$

Taking the required expectations

$$E\{Z\} = m_z = m_{y_1} + E \ln(1 + e^w)$$

$$= m_{y_1} + \int_{-\infty}^{+\infty} \ln(1 + e^w) f(w) dw. \tag{47}$$

As discussed earlier, the integral is broken up into the intervals $(-\infty, 0)$ and $(0, \infty)$. Upon application of the series (36) and the averaging (42) and (43), we obtain the result

$$m_z = m_{y_1} + m_w \Phi\left(\frac{m_w}{\sigma_w}\right) - \frac{\sigma_w}{\sqrt{2\pi}} e^{-m_w^2/2\sigma_w^2}$$

$$+ \sum_{k=1}^{\infty} C_k e^{k^2\sigma_w^2/2}\left[e^{km_w}\Phi\left(\frac{-m_w - k\sigma_w^2}{\sigma_w}\right)\right. \tag{48}$$

$$\left. + e^{-km_w}\Phi\left(\frac{m_w - k\sigma_w^2}{\sigma_w}\right)\right].$$

All the terms, except m_{y_1}, are lumped together and denoted by $G_1(\sigma_w, m_w)$:

$$m_z = m_{y_1} + G_1(\sigma_w, m_w). \tag{49}$$

The function G_1 is shown graphically in Fig. 5.

The second moment is decomposed as

$$E(Z^2) = E\{[Y_1 + \ln(1 + e^w)]^2\}$$

$$= \sigma_{y_1}^2 + m_{y_1}^2 + E[2Y_1 \ln(1 + e^w)] + E[\ln^2(1 + e^w)]. \tag{50}$$

First, consider the cross term

$$A = E[2Y_1 \ln(1 + e^w)], \tag{51}$$

where Y_1 and $w = Y_2 - Y_1$ are jointly Gaussian with correlation coefficient

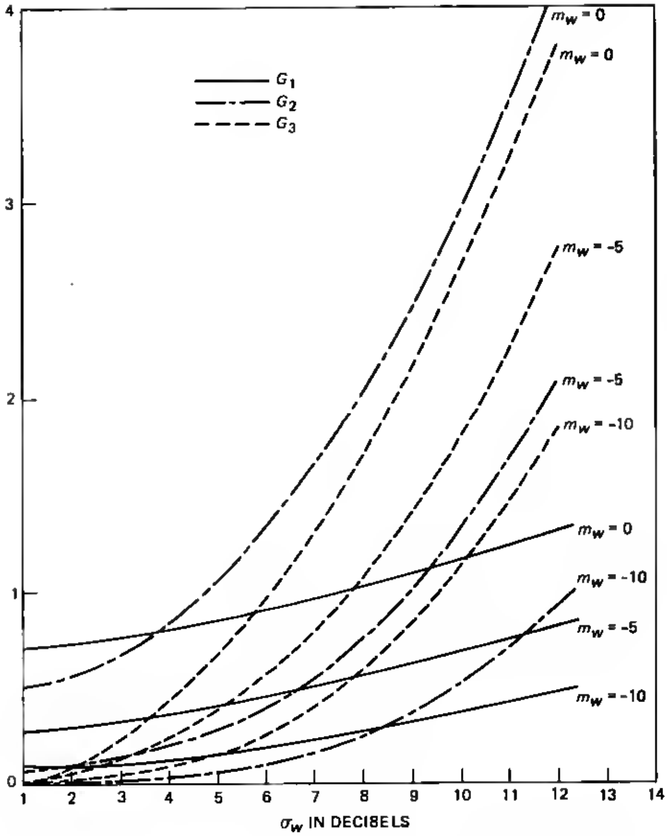


Fig. 5—The plots of G_i's as a function of σ_w with m_w as a parameter.

$$\begin{aligned}\rho &= E\left[\frac{(Y_1 - m_{y_1})(w - m_w)}{\sigma_{y_1}\sigma_w}\right] \\ &= -\frac{\sigma_{y_1}}{\sigma_w} \\ &= -\sigma_{y_1}/\sqrt{\sigma_{y_1}^2 + \sigma_{y_2}^2}\ .\end{aligned} \tag{52}$$

We utilize the conditional expectation property of jointly Gaussian random variables (see Ref. 17, Section 7-5):

$$E[(Y_1 - m_{y_1})|(w - m_w)] = \frac{\rho\sigma_{y_1}}{\sigma_w}(w - m_w) \tag{53}$$

or

$$E[(Y_1|w)] = m_{y_1} - \rho^2(w - m_w).$$

Then, (51) becomes

$$
\begin{aligned}
A &= E[2Y_1 \ln(1 + e^w)] \\
&= 2E_w[\ln(1 + e^w) E_{Y_1} | w(Y_1)] \\
&= 2E_w\{\ln(1 + e^w)[m_{y_1} - \rho^2(w - m_w)]\}.
\end{aligned} \tag{54}
$$

For convenience, we denote the first and second terms by A_1 and A_2, respectively:

$$A = A_1 - A_2. \tag{55}$$

Since

$$
\begin{aligned}
A_1 &= 2m_{y_1} E[\ln(1 + e^w)] \\
&= 2m_{y_1}(m_z - m_{y_1})
\end{aligned} \tag{56}
$$

from (47), we need only consider the second term A_2,

$$A_2 = 2\rho^2 E[(w - m_w) \ln(1 + e^w)]. \tag{57}$$

Integrate A_2 by parts:

$$
\begin{aligned}
A_2 &= 2\rho^2 \int_{-\infty}^{+\infty} (w - m_w) \ln(1 + e^w) f(w) dw \\
&= 2\rho^2 \sigma_w^2 \int_{-\infty}^{+\infty} \frac{e^w}{1 + e^w} f(w) dw.
\end{aligned} \tag{58}
$$

For negative w, we use the expansion

$$\frac{e^w}{1 + e^w} = \sum_{k=0}^{\infty} (-1)^k e^{(k+1)w} \tag{59}$$

to obtain

$$
\begin{aligned}
A_2' &= 2\rho^2 \sigma_w^2 \int_{-\infty}^{0} \frac{e^w}{1 + e^w} f(w) dw \\
&= 2\rho^2 \sigma_w^2 \sum_{k=0}^{\infty} (-1)^k e^{m_w(k+1) + (k+1)^2 \sigma_w^2/2} \\
&\qquad \Phi\left[\frac{-m_w - \sigma_w^2(k + 1)}{\sigma_w}\right].
\end{aligned} \tag{60}
$$

For the other part of (58), we have

$$
\begin{aligned}
A_2'' &= 2\rho^2 \sigma_w^2 \int_{0}^{\infty} \frac{e^w}{1 + e^w} f(w) dw \\
&= 2\rho^2 \sigma_w^2 \int_{0}^{\infty} \frac{1}{1 + e^{-w}} f(w) dw
\end{aligned}
$$

$$= 2\rho^2\sigma_w^2 \int_0^\infty \sum_{k=0}^\infty (-1)^k e^{-kw}(f(w)dw$$

$$= 2\rho^2\sigma_w^2 \sum_{k=0}^\infty (-1)^k e^{-km_w+k^2\sigma_w^2/2}$$

$$\Phi\left[\frac{m_w - \sigma_w^{2k}}{\sigma_w}\right]. \tag{61}$$

Collecting the results for the cross term,

$$A = A_1 - A_2 = A_1 - (A_2' + A_2'')$$

$$= A_1 - 2\rho^2 G_3(\sigma_w, m_w)$$

$$= 2m_{y_1} G_1(\sigma_w, m_w) - 2\rho^2 G_3(\sigma_w, m_w). \tag{62}$$

The remaining term in $E(Z^2)$ to evaluate is the last expression in (50). We denote it by $G_2(\sigma_w, m_w)$.

$$G_2(\sigma_w, m_w) = E[\ln^2(1 + e^w)]$$

$$= \int_{-\infty}^0 \ln^2(1 + e^w)f(w)dw + \int_0^\infty \ln^2(1 + e^w)f(w)dw$$

$$= B_1 + B_2. \tag{63}$$

For the integration over the negative real line, we can use the expansion given by (37):

$$B_1 = \sum_{k=1}^\infty b_k \int_{-\infty}^0 e^{(k+1)w} f(w)dw$$

$$= \sum_{k=1}^\infty b_k e^{[(k+1)m_w+(k+1)^2\sigma_w^2/2]}$$

$$\Phi\left[\frac{-m_w - \sigma^2(k+1)}{\sigma_w}\right]. \tag{64}$$

The expression B_2 is rewritten as:

$$B_2 = \int_0^\infty \{\ln[e^w(1 + e^{-w})]\}^2 f(w)dw$$

$$= \int_0^\infty [\ln^2(1 + e^{-w}) + 2w\ln(1 + e^{-w}) + w^2]f(w)dw$$

$$= B_5 + B_4 + B_3. \tag{65}$$

The expressions for B_3 and B_5 are relatively straightforward:

$$B_3 = \int_0^{\infty} w^2 f(w)\,dw$$

$$= m_w^2\left[1 - \Phi\left(\frac{-m_w}{\sigma_w}\right)\right]$$

$$+ \sqrt{2/\pi}\, m_w \sigma_w e^{-m_w^2/2\sigma_w^2}$$

$$+ \sigma^2\left[1 - \Phi\left(\frac{-m_w}{\sigma_w}\right)\right] \frac{-m_w\sigma_w}{\sqrt{2\pi}}\, e^{-m_w^2/2\sigma_w^2} \tag{66}$$

$$B_5 = \int_0^{\infty} \ln^2(1 + e^{-w}) f(w)\,dw$$

$$= \int_{-\infty}^{0} \ln^2(1 + e^{w}) f(-w)\,dw$$

$$= \int_{-\infty}^{0} \sum_{k=1}^{\infty} b_k e^{(k+1)w} f(-w)\,dw$$

$$= \sum_{k=1}^{\infty} b_k e^{-(k+1)m_w + (k+1)^2\sigma_w^2/2}$$

$$\Phi\left[\frac{m_w - \sigma_w^2(k+1)}{\sigma_w}\right]. \tag{67}$$

The final term to consider is

$$B_4 = \int_0^{\infty} 2w \ln(1 + e^{-w}) f(w)\,dw$$

$$= -2 \sum_{k=1}^{\infty} C_k \int_{-\infty}^{0} w e^{kw} f(-w)\,dw. \tag{68}$$

Complete the square for the expression $e^{kw}f(-w)$ and integrate to obtain

$$B_4 = -2 \sum_{k=1}^{\infty} C_k e^{(-m_w k + k^2\sigma_w^2/2)}\left[m_k \Phi\left(\frac{-m_k}{\sigma_w}\right) - \frac{\sigma_w}{\sqrt{2\pi}}\, e^{-m_k^2/2\sigma_w^2}\right], \tag{69}$$

where

$$m_k = -m_w + k\sigma_w^2. \tag{70}$$

Collecting the terms that make up $G_2(\sigma_w, m_w)$ in (63), we have

$$G_2(\sigma_w, m_w) = B_1 + B_2 = B_1 + (B_3 + B_4 + B_5) \tag{71}$$

as given by eqs. (64), (66), (69), and (67).

To summarize, the second moment is given by the expression

$$E(Z^2) = \sigma_{y_1}^2 + m_{y_1}^2 + 2\, m_{y_1} G_1(\sigma_w, m_w) - 2\rho^2 G_3(\sigma_w, m_w) + G_2(\sigma_w, m_w), \tag{72}$$

and, finally, the variance is

$$\sigma_z^2 = E(Z^2) - m_z^2 = \sigma_{y_1}^2 - G_1^2(\sigma_w, m_w) - 2\rho^2 G_3(\sigma_w, m_w) + G_2(\sigma_w, m_w). \tag{73}$$

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